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TECHNICAL NOTES
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS.

No. 205

THE LOGARITHMIC POLAR CURVE - ITS THEORY AND APPLICATION
TO THE PREDETERMINATION OF AIRPLANE PERFORMANCE.

By Val. Cronstedt.

October, 1924.

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Introduction

The logarithmic polar curve has for several years been used by the most prominent aerodynamical laboratories as well as by airplane manufacturers in Europe.

The vast possibilities of the method, when once thoroughly mastered and properly used, and the saving of time and expense involved in the analytical treatment of performance, amply justify its use. Any one who familiarizes himself with the method will, within a short time, find a large number of applications, not mentioned in this note, which, however, will come up in connection with airplane design. As examples, some additions to the original method which I have made myself and which may increase the usefulness of the method for performance estimation, are described and their construction shown.

To show more clearly the practical application of the polar curve, a series of examples are appended hereto with suggestions for solutions. I am indebted to Mr. Elliott G. Reid of the N.A.C.A. Langley Memorial Aeronautical Laboratory for his thorough checking of the manuscript and also for the valuable suggestions he has made.

In making up the appended chart, engineering units have been used throughout, as they are more convenient to use in a design office, giving directly the results in easily visualized units.

In working on the problems connected with aerodynamics and related subjects, graphical solutions are often used due to the facility with which the designer can obtain from them his data and thenceforth make suitable changes in his design, if needed, to meet certain requirements. To show graphically the relations between the air forces acting on an airplane, its velocity, angle to the relative wind (angle of attack), climbing ability, etc., is therefore desirable - and several attempts have been made to obtain a satisfactory graphical method.

One method was demonstrated by Eiffel in his work "La Resistance de L'air et L'aviation" and mentioned in his later works, among others, "Nouvelles Recherches sur la Resistance de L'air et L'aviation," and in a modified form in "Zeitschrift für Flugtechnik und Motorluftschiffahrt," by Dr. E. Everling, but they are lacking in several details. The last named, especially, has the undesirable feature that certain values necessary for the design cannot be solved, but only approximately estimated. One method, based on vector algebra, which has great merits and shows in a very neat way the variations of the performance properties of an airplane subjected to different arrangements of power plant, supporting surfaces,

etc., has been suggested by Rith, of the Eiffel Laboratory at Auteuil, France.

I have attempted by this note to outline the theory and to show some of the possibilities of the Rith method, generally known as the "logarithmic polar curve" for the predetermination of airplane performance, and also to show some modifications required by more recent conceptions of performance.

I. The Theory of the "Logarithmic Polar Curve."

In the study of airfoil profiles, several organizations make use of a polar curve in Cartesian coordinates which must not be confused with the polar curve in logarithmic coordinates, the latter being used solely for the study of complete airplanes. The different forces acting on an airplane may be replaced by one force R which in turn may be divided into the two components D , or drag, parallel to the direction of flight, and L , or lift, perpendicular to D . We can obtain the common polar by plotting the values of L and D , for unit velocity, along the axes of a Cartesian coordinate system. Herein a line drawn from the origin to any point on the curve represents the direction and magnitude of the resultant air force R .

If, instead of plotting the values of L and D themselves, we plot their logarithms as Rith has suggested, we obtain a continuous curve, the logarithmic polar curve.

To plot the polar curve, we must, therefore, know the values of these forces for different angles of attack. These values may be ob-

tained by wind tunnel measurements or analytically. When they have been obtained, one way or another, we plot their logarithms along the respective axes in a logarithmic coordinate system. With the help of the graduations found on a slide rule, or better, by using logarithmically graduated plotting paper, this can easily be done. For routine work it is advisable to have blanks made up similar to Fig. 14 of Appendix III.

Let us adopt the following notation:

P = Thrust power in lb.-ft. per second.

W = Weight of airplane in pounds.

S = Wing area in sq.ft.

V = Velocity in feet per second.

$C_L = \frac{L}{Sq}$ (Lift coefficient, absolute).

$C_D = \frac{D}{Sq}$ (Drag coefficient, absolute).

wherein

L = Lift force in pounds.

D = Drag force in pounds.

$q = \rho \frac{V^2}{2}$ = Dynamic pressure in lb. per sq.ft.

ρ = Mass density of air (slugs per cu.ft.)

The two fundamental aerodynamic equations are:

$$P = C_D SqV = \frac{\rho}{2} C_D SV^3 \quad (1)$$

$$W = C_L Sq = \frac{\rho}{2} C_L SV^2 \quad (2)$$

wherein

$C_D Sq$ = Drag

$C_L Sq$ = Lift

If the following values be substituted:

$$\frac{\rho}{2} C_D S = K'_x \text{ (Drag in lb. at } V = 1 \text{ ft./sec.)}$$

$$\frac{\rho}{2} C_L S = K'_y \text{ (Lift in lb. at } V = 1 \text{ ft./sec.)}$$

the original equations become:

$$P' = K'_x V'^3 \quad (3)$$

$$W = K'_y V'^2 \quad (4)$$

Writing (3) and (4) in logarithmic form:

$$\log P' = \log K'_x + 3 \log V' \quad (5)$$

$$\log W = \log K'_y + 2 \log V' \quad (6)$$

and transposing, we have

$$\log K'_x = \log P' - 3 \log V' \quad (7)$$

$$\log K'_y = \log W - 2 \log V' \quad (8)$$

These equations define the logarithmic polar curve.

Considering $\log K'_x$ and $\log K'_y$ as abscissa and ordinate, respectively, the above equations show that each has a component expressed in terms of $\log V'$. The components of $\log K'_x$ and $\log K'_y$ are plotted, diagrammatically, in Fig. 1.* Now as V' has the same value in both equations, i.e., the equations are simultaneous, and as the components in $\log V'$ bear a constant relation to each other, the addition of a third axis to the diagram makes it possible to plot the resultant of the two components, $-3 \log V'$ and $-2 \log V'$, directly. This "velocity axis" has the slope $2/3$ and is also graduated logarithmically. Its modulus is

* See Page 6 for footnote.

$$ab = \sqrt{3^2 + 2^2} K'_x \text{ (or } K'_y)$$

= 3.605 times the modulus of the K'_x and K'_y axes.

To make the chart more directly applicable, the scales are graduated doubly, i.e., along the K_x scale we have P in horsepower, along K_y there is a W scale reading in pounds, and the V scale may be graduated both in M.P.H. and ft. per second, if desired.

When a reference value (V_0) is chosen on the V axis, the relations between the other double scales become fixed. (It is worthy of note that only the directions, and not the positions of the logarithmic scales are essential to the use of the chart.) For practical work I have found it advisable to use a value $V_0 = 200$ M.P.H.

As the M.P.H. scale on the V axis will be most generally used

* It will be noted that $-3 \log V'$ and $-2 \log V'$ have been plotted as positive quantities, if referred to the K'_x and K'_y axes. The reason for doing this becomes evident from an examination of equations (3) and (4). There one will see that K'_x varies with $1/V'^3$ and K_y with $1/V'^2$. It then becomes necessary to graduate the V axis so that this inverse variation will hold. This is done by giving the axis the slope $2/3$, a modulus 3.605 times that of the K_x and K_y axes and making its positive sense toward the left and downward.

That this process is the rational one may be confirmed by the following case. Let us suppose that the polar (Fig. 1) intersects the V axis at the point $(\log P', \log W)$. The V components are then zero and, under these conditions, the airplane represented by this polar will maintain level flight at the velocity V_0 , chosen as reference on the V axis. Now let us suppose that P' and W are increased in such proportion that the point $(\log P_1, \log W_1)$ is on the V axis, but to the right and above the point first considered. In this case, the resultant of the velocity components must be directed downward and to the left. That this vector must represent a velocity greater than V_0 is evident from the fact that both conditions of flight involve the same attitude and, consequently, the same values of C_L and L/D . As in the second a greater weight is being supported, the velocity of flight must be greater to provide the additional lift necessary.

we must transform equations (3) and (4) accordingly. Also we will now express P in horsepower rather than lb.-ft. per second. The equations then become

$$550 P = \frac{K_X}{\left(\frac{5280}{3600}\right)^2} \left(V \frac{5280}{3600}\right)^3 \quad (9)$$

$$\text{or } K_X = \frac{550 P \times 3600}{5280 V^3} = \frac{375 P}{V^3} \quad (9a)$$

$$\text{and } W = \frac{K_Y}{\left(\frac{5280}{3600}\right)^2} \left(V \frac{5280}{3600}\right)^2 \quad (10)$$

$$\text{or } K_Y = \frac{W}{V^2} \quad (10a)$$

wherein

P is power in HP

K_X is drag in lb. at one M.P.H.

V is velocity in M.P.H.

K_Y is lift in lb. at one M.P.H.

Arbitrarily assuming $P = 100$ HP and $W = 1000$ lb., we now solve (9a) and (10a) and find

$$K_X = 0.00468$$

$$\text{and } K_Y = 0.025$$

Thus, $K_X = 0.00468$ corresponds to 100 HP and

$K_Y = 0.025$ to 1000 lb. at 200 M.P.H.

To still more increase the value of the system we will add two more scales.

The first scale, or rather, scale system, is designed to obtain the thrust horsepower when propeller efficiency η , and engine

horsepower P_m are known. This scale system is originated by the writer, and its application is shown in Fig. 2.

To obtain thrust horsepower when engine horsepower and propeller efficiency are known, draw a line parallel to the lines shown on the appended graph from engine horsepower at a . A line from propeller efficiency at b parallel to the X-axis intersecting the oblique line at d will give thrust horsepower at c . The graduation on the propeller efficiency scale is naturally logarithmic, with a modulus independent of all the other scales, and can be given any value suiting the individual user. This system is introduced for the first time here and has worked out very well in practice. The influence of propeller efficiency on the various performance factors can quickly be shown and the correct propeller efficiency chosen for each condition.

The other scale is a size scale. If we wish to investigate the performance of an airplane which is geometrically similar to the one whose polar curve is known, we need not draw a new curve but may use the size scale for this purpose. Our new axis will be graduated in terms of the ratio between corresponding linear dimensions of the two airplanes.

Assume that we want to increase the linear size of an airplane n times. The supporting area in the new case will be

$$S_1 = n^2 S$$

Then equations (3) and (4) may be written

$$P = n^2 K_X V^3 \quad (11)$$

$$W = n^2 K_Y V^2 \quad (12)$$

or logarithmically

$$\log K_x = \log P - 3 \log V - 2 \log n \quad (13)$$

$$\log K_y = \log W - 2 \log V - 2 \log n \quad (14)$$

With the same deduction as before, we find the slope of the new axis to be $2/2 = 1$ (Fig. 3).

The logarithmic modulus of this axis is therefore

$$\sqrt{2^2 + 2^2} = \sqrt{8} = 2.828 \text{ times the modulus of the}$$

K_x and K_y axes.

This axis is also graduated as an altitude scale on the basis of the following principle:

If angle of attack and velocity remain constant, the forces on an airplane vary directly with air density. Therefore we may write

$$P_z = \frac{\rho_z}{\rho_0} K_x V^3 \quad (15)$$

$$W_z = \frac{\rho_z}{\rho_0} K_y V^2 \quad (16)$$

wherein the subscript z denotes quantities existing at the altitude at which air density is ρ_z , and ρ_0 is the density at ground level. Equations (15) and (16) may be put into logarithmic form as

$$\log K_x = \log P_z - 3 \log V - \log \frac{\rho_z}{\rho_0} \quad (17)$$

$$\log K_y = \log W_z - 2 \log V - \log \frac{\rho_z}{\rho_0} \quad (18)$$

Here, as in the case of the n scale, we have equal components of ordinate and abscissa, but this time in terms of the density ratio.

A "density ratio axis" will then have the same slope as that of the n scale, i.e., +1, and a modulus $\sqrt{2}$ times that of the K_x and K_y scales (See Fig. 4).

As it is more convenient to work with altitude than density ratio, a logarithmic density ratio scale has been made up, the equivalent density ratios noted thereon and the scale of Z in thousands of feet used on the chart.

(The density ratios used are those adopted by the U.S. Navy Bureau of Aeronautics as "Standard Atmosphere." A table of these ratios is given in Appendix I.)

It must be borne in mind that the altitude scale cannot be used directly, but must be used in connection with a method of correcting the available engine power for the influence of the reduced density at an altitude.

Let us first assume that engine power varies directly with density ratio, i.e., $P \sim \frac{\rho_z}{\rho_0}$. To represent this variation we use the construction shown in Fig. 5. The power at ground level is represented by the vector AB . To find the power at any altitude, we erect a perpendicular at B and the length of a horizontal line, such as CD , from the Z axis to this perpendicular represents the power at the altitude C .

In most modern aviation engines, however, the power decreases somewhat more rapidly than the density ratio and the general average seems to be best expressed by

$$HP \sim \left(\frac{\rho_z}{\rho_0} \right)^{1.1}$$

Therefore, if the correcting line has a slope of 1.1 with respect to the Z-axis, the variation is taken care of. To this end, the scale in the lower right corner of the chart has been provided and is graduated directly in terms of the power of the density ratio assumed to govern the variation of engine power. This scale is extended in both directions so that the altitude performance of all engines, whether supercharged, "over-dimensioned" or not attaining the 1.1 ratio may be followed. The methods employed in the use of this system are self-explanatory and can be readily followed on the complete example worked out in Fig. 14, Appendix III.

II. The Practical Application of the Polar Curve.

To show some of the methods of applying the polar curve to practical problems, a series of problems will be given below and methods suggested for solving the same.

(1) Given: Airplane gross weight W
Engine Horsepower P

To find: Velocity V and angle of attack α .

We assume that the polar curve for the airplane is known and is the one shown in Fig. 6.

Along the respective axis are plotted W and P and the point a obtained. From a we draw a line parallel to the V -axis until this line intersects our polar curve. This happens as we see, in this case, at two points b and c . This shows that this airplane can sustain flight at two different angles using the same engine

power. The angles of attack are of course the ones corresponding to b and c .

The velocities are represented by the vectors ab and ac respectively; their numerical values are obtained by laying off, from V_0 , the vectors in the direction in which they are drawn from a .

(2) Given: Airplane gross weight

To find: Minimum power for level flight and corresponding angle of attack and velocity.

The construction is given in Fig. 7. A parallel to the V -axis is drawn tangent to the polar curve. The tangent intersects a horizontal line from the point W at a . The vector Wa represents minimum power, ab , the velocity, and the angle of attack is determined from the position of b on the polar.

This is also the condition for maximum duration as minimum power corresponds to minimum gross fuel consumption.

(3) Given: Airplane gross weight

Velocity

To find: Required engine power and angle of attack.

As in Fig. 8, the known values V and W are marked along respective axes and hereby we obtain point a . Thereafter, a line is drawn parallel with the P -axis until it intersects our curve. The angle of attack corresponding to the point b is the required flying angle and $a-b$ is the required horsepower.

If we move the line $a-b$ parallel with itself we are reaching as a limit, the point where the line is a tangent to the polar curve.

This gives us the maximum weight of the airplane for a given velocity and so we have gone over to problem 4.

(4) Given: Velocity

To find: Maximum possible gross weight.

As shown in Fig. 9, if we draw a tangent line parallel to the P-axis, it will intersect at a line drawn parallel to the W-axis from the given velocity and we have hereby our W maximum.

(5) Given: Engine power and velocity

To find: Gross weight and angle of incidence.

As in Fig. 10, the values P and V are plotted along their respective axes, thereby obtaining the point a. From a is drawn a line parallel to the W-axis until it intersects the curve. The angle of attack corresponding to point b is the angle sought and the distance a-b the maximum possible gross weight with the given power.

Keeping the engine power at the same value but moving a-b to the left we note the velocity increases. With the help of Fig. 11, we can therefore solve problem 6.

(6) Given: Engine power

To find: Maximum horizontal velocity and corresponding optimum weight.

Draw a line parallel to the W-axis and tangent to the polar. Its intersection with the line of the velocity vector at d fixes V and d_c represents the optimum weight.

(7) To find: Angle of attack and L/D ratio for flattest glide.

The only forces acting on an airplane during a steady glide are weight and total air reaction, propeller thrust being zero. The two forces are, necessarily, collinear, equal, and opposite. It is known that the lift vector forms the same angle with that of the total air force as does the horizontal with the line of flight. Let us denote this angle by γ . Then the best glide possible is that in which γ is minimum. Now, as $\gamma = \tan^{-1} \frac{K_x}{K_y}$, this condition is attained when the ratio $\frac{K_x}{K_y}$ is minimum or the L/D ratio is maximum.

Let us write $\frac{K_x}{K_y} = C$.

$$\text{Then } \log K_x = \log K_y - \log C \quad (19)$$

This is evidently the equation of a line parallel to the line $\log K_x = \log K_y$, whose slope is one, and at a distance C above it. To reach $\frac{K_x}{K_y}$ minimum, C must also be minimum. From this fact and equation (19) we see that as C decreases, the line of slope = 1 (45°) will move toward the left. Therefore, we will find $\frac{K_x}{K_y}$ minimum, or L/D maximum at the point of tangency of a 45° line with the polar curve. This defines the angle of attack for best glide.*

To find the L/D ratio of this or any other point of the polar, a scale has been provided as shown in Fig. 12. As $-\log C$ is taken in the direction of the K_y axis and is a first degree term in the equation, the C or L/D scale has the same modulus as that of K_y . We take as reference value, $\left(\frac{L}{D}\right)_0$, any point of the chart for which

* It is known that any straight line which passes through the origin of a system of Cartesian coordinates will appear as a straight line of slope = +1 (45°) when plotted to logarithmic coordinates. As the value of L/D maximum is obtained by drawing a polar tangent to the Cartesian K_y vs. K_x curve, the 45° tangent is the logarithmic representation of this line.

$K_x = K_y$, i.e., at which $\frac{L}{D} = 1$.

To find the value of the L/D ratio for any point of the polar it is only necessary to draw a line at 45° through the point and note its intersection with the L/D scale.

Problems Involving Climb.

To solve problems involving climb characteristics with the aid of the logarithmic diagram, we must make the following assumptions:*

- (a) Engine speed remains constant regardless of altitude.
- (b) Engine power varies with a given power of the density ratio.
- (c) In climb the engine develops only 90% full power.

In justification of the first assumption there is the fact that in climb tests the variation of engine speed from sea level to ceiling is very small, the average drop being about 5 per cent. With regard to (b), it has been found that the proper exponent for the density ratio is slightly different for different engines but a good average value is 1.1. The third assumption is less reliable than the other two because it depends upon so many factors, the most important being propeller characteristics. While 90% is a fair value for the average airplane which has a considerable speed range and good climb, the selection of this factor for any new airplane of unusual characteristics will require the use of sound judgment and may vary considerably between different types.

* If it is chosen to work out altitude problems "step by step" these assumptions may naturally be disregarded.

To obtain the ceiling of our airplane, we use the construction shown in Fig. 13. Knowing gross weight, maximum power and propeller characteristics, we locate the point A. Drawing the tangent CD, which is the velocity vector for horizontal flight at minimum power, we intercept the distance CA, which represents the power available for climb. Now we draw the power-correcting line through A, using the scale system in the lower right corner of the chart to determine its slope.

The distance between the tangent defining minimum power and the power-correcting line represents the surplus power available for climb at the altitude read from the auxiliary scale CZ, and we see that at B, all the available power is required to maintain level flight. This, then, is the absolute ceiling and the airplane flies at the speed BD.

To compute the rate of climb, we have only to solve the equation

$$\text{Climb (ft./min.)} = \text{HP}_{\text{ex}} \times 33,000/W$$

wherein HP_{ex} is the surplus power available. In scaling off this quantity it is essential that the ends of the horizontal line representing surplus power be projected parallel to the Z-scale onto the sea level power vector. The values of thrust power available and thrust power required may then be read by direct projection onto the thrust power axis.

The location of service ceiling follows directly from this process.* We merely determine the surplus power necessary to give climb

* Service ceiling is defined as that altitude at which the maximum climbing velocity attainable is 100 ft./min.

of 100 ft./min. and locate the altitude at which this excess exists.

The Problem of Speed at Altitude.

We have developed the solution for maximum speed at sea level and now have a close approximation for ceiling. Using these quantities, we may now solve for maximum and minimum speeds as well as the speed of best climb for any altitude.

It is known that the R.P.M. of the engine, for maximum level speed, will decrease with altitude and at ceiling will have the same value as those for minimum and best climbing speeds, the three being coincident. Also, as V/ND decreases, the propeller efficiency for maximum speed will approach that in climb and the two will become identical at ceiling. Then we may represent the maximum thrust power available at any altitude by a line connecting the points c and y , as shown in Fig. 14, Appendix III, and the velocity vectors representing $V_{\max}$ for all altitudes will originate in this line.

The speed of best climb at any altitude is easily found as this is the speed of minimum power required for level flight.

The minimum speed, being that corresponding to maximum lift coefficient, will be easily found for all altitudes until we approach ceiling. When such solutions are desired, care must be taken to have the velocity vector originate within the limits of available power. An example of this kind is shown in the solution for minimum speed at 30,000 feet in the problem of Appendix III.

Appendix I.

U. S. Navy, Bureau of Aeronautics, Standard Atmosphere.

Altitude, feet.	Density, ratio
0	1.000
1000	.9710
2000	.9428
3000	.9152
4000	.8881
5000	.8617
6000	.8358
7000	.8106
8000	.7860
9000	.7619
10000	.7384
12000	.6931
14000	.6500
16000	.6089
18000	.5699
20000	.5328
22000	.4975
24000	.4641
26000	.4324
28000	.4024
30000	.3741
32000	.3472
34000	.3219
36000	.2980
38000	.271
40000	.245

Appendix II.

Characteristics of Airplane Analyzed in Appendix III.

Gross weight $W = 4800$ lb.

Engine power $P = 700$ HP

Engine power remains constant to 5000 ft. and then varies as the density ratio to 1.1 power.

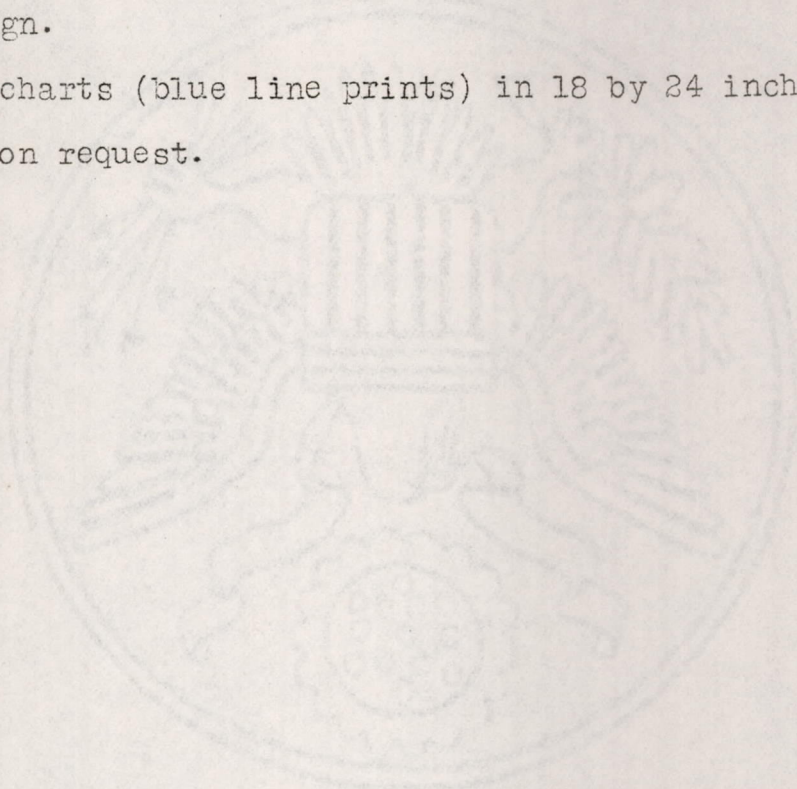
Lift and drag of full size airplane at 1 M.P.H.

Angle of attack	K_y	K_x
-2	.009	.0611
0	.270	.0603
+2	.436	.0630
+4	.611	.0700
6	.782	.0823
8	.935	.0960
10	1.098	.116
12	1.240	.139
14	1.368	.162
16	1.478	.191
18	1.540	.226
20	1.520	.268

Notes to Appendix III.

In Appendix III (Fig. 14), a complete example has been worked out according to the processes detailed in the preceding pages. The polar curve shown in this example was obtained by wind tunnel measurements and refers to a two-seater observation airplane of recent design.

Blank charts (blue line prints) in 18 by 24 inch size may be obtained upon request.



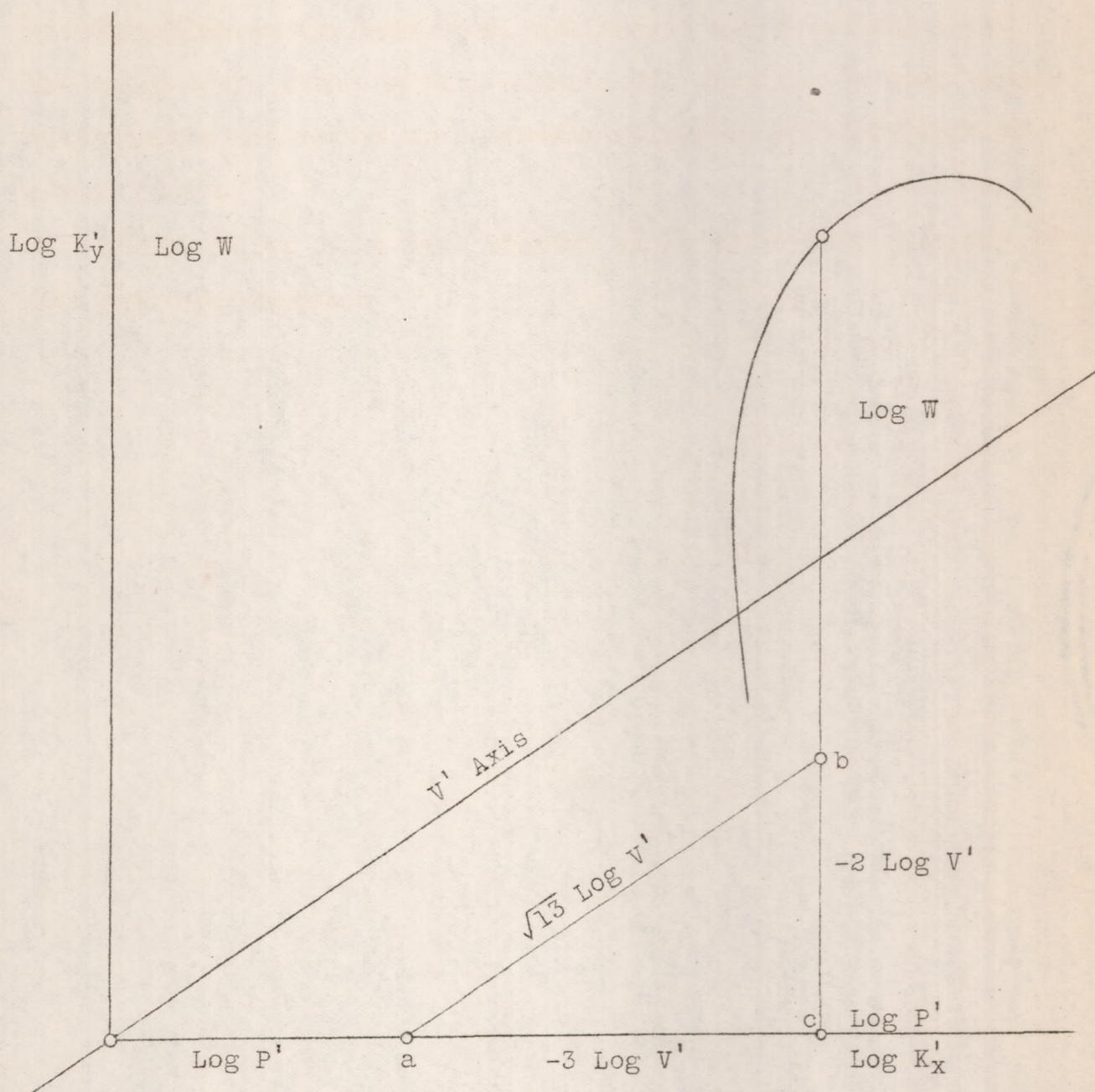


Fig.1

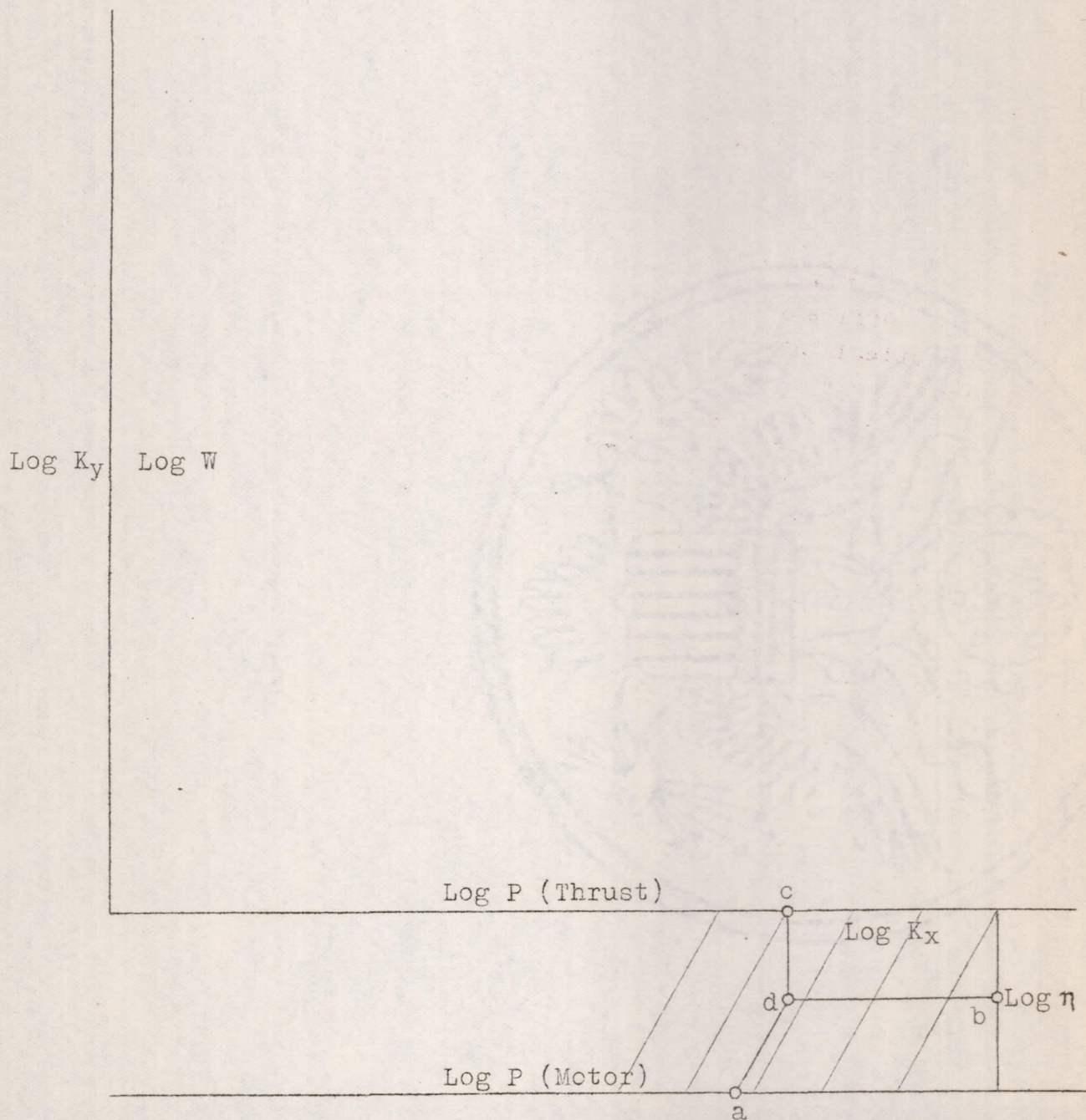


Fig.2

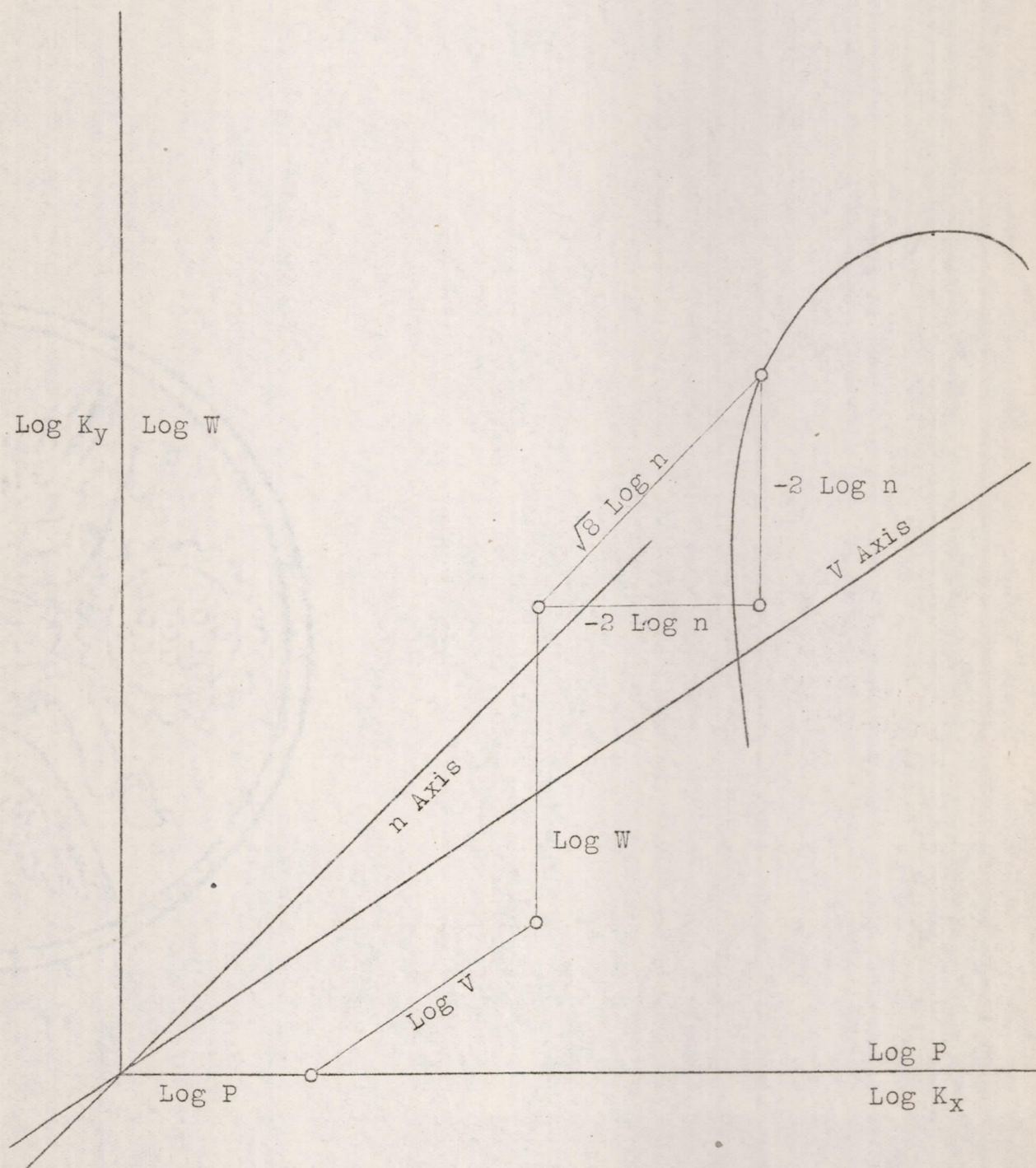


Fig.3

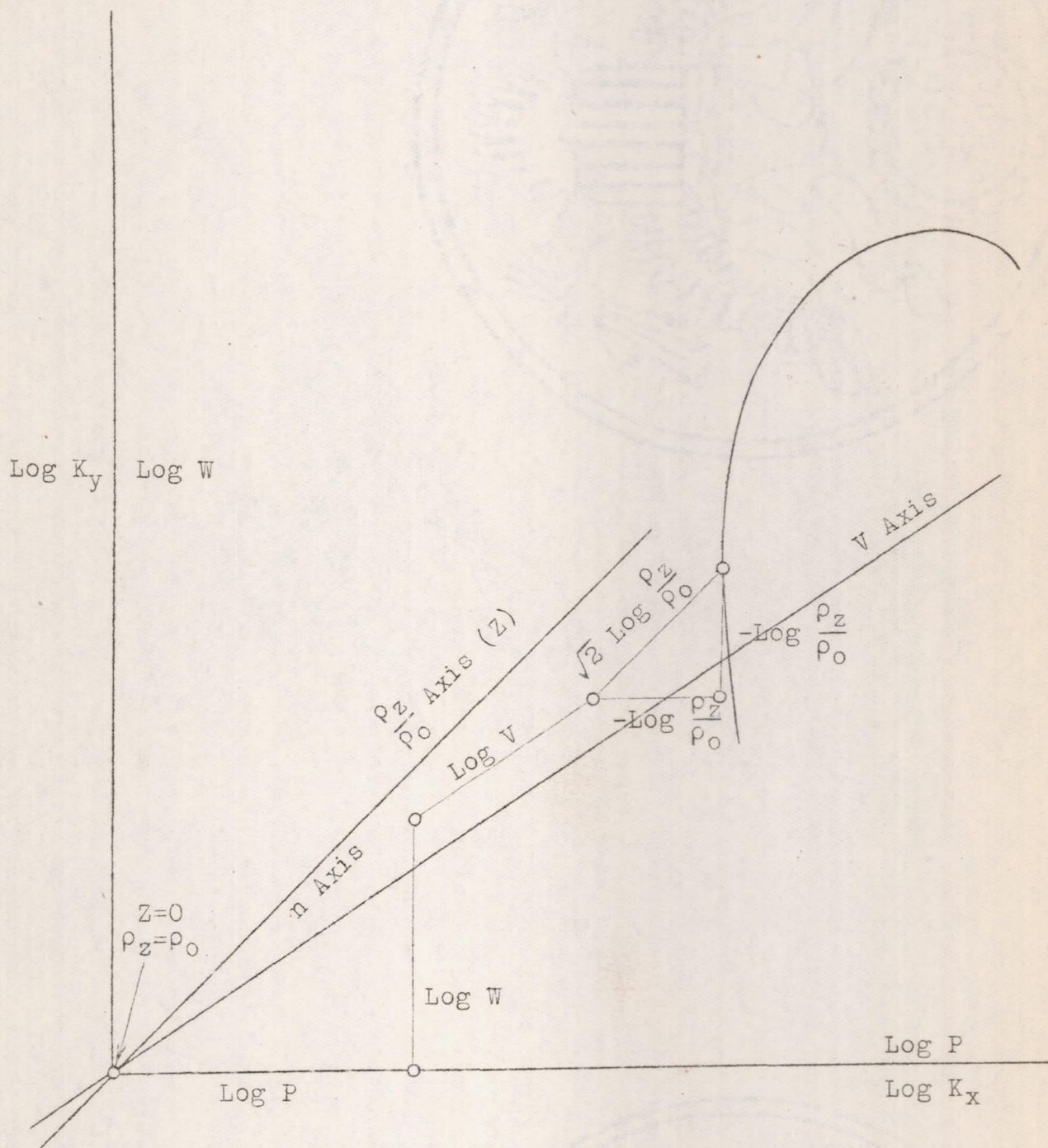


Fig.4

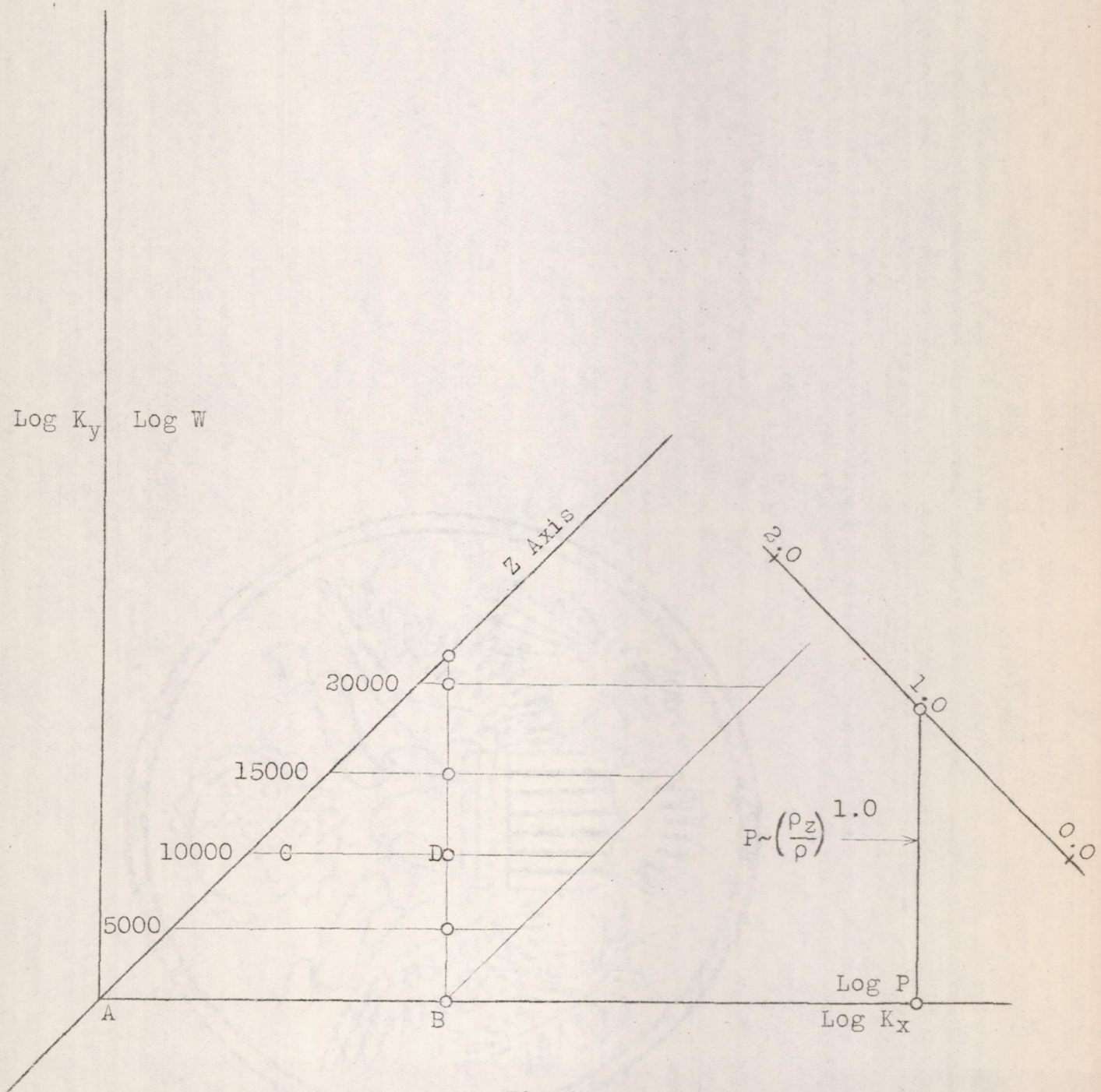
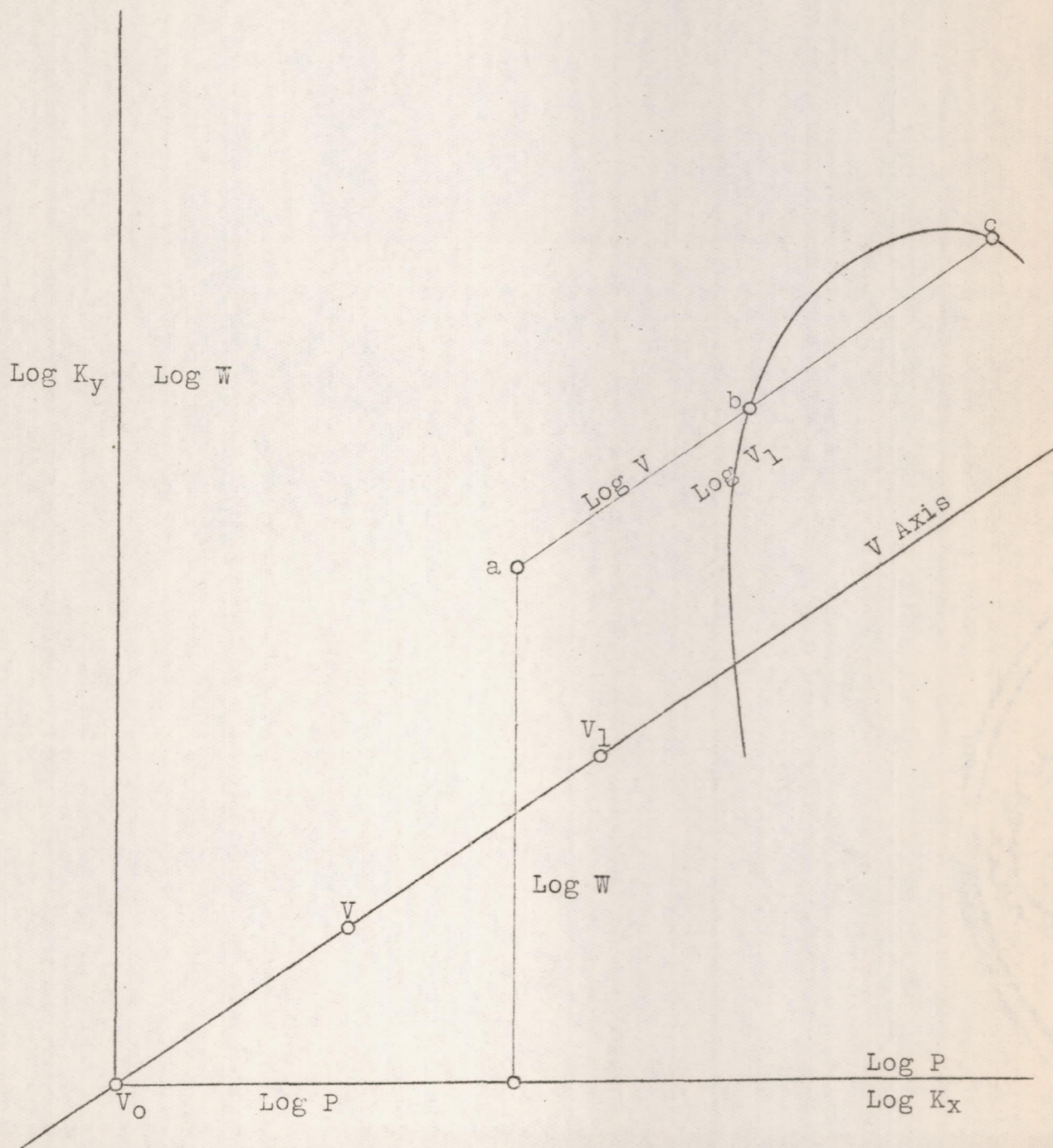


Fig.5



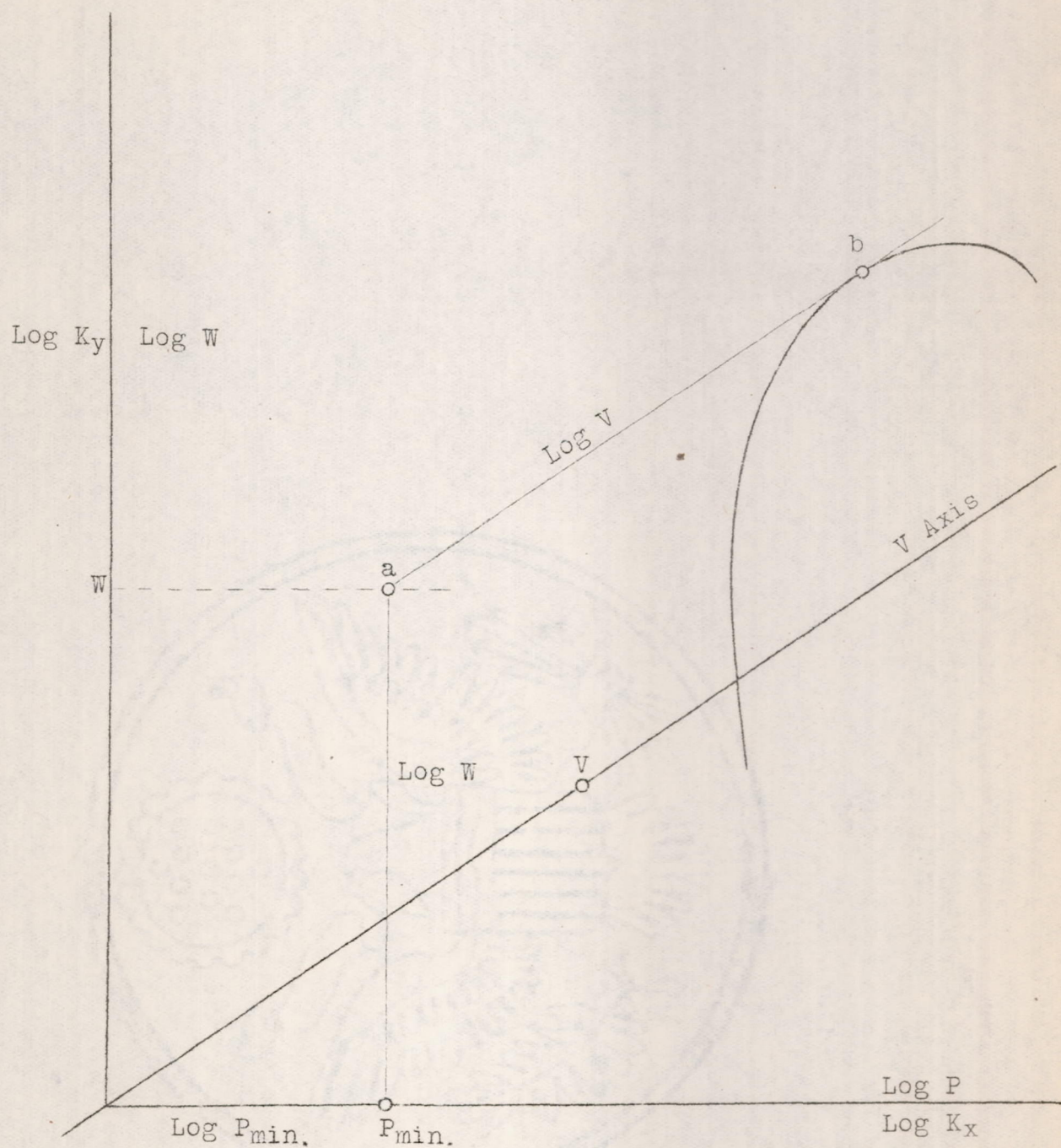


Fig.7

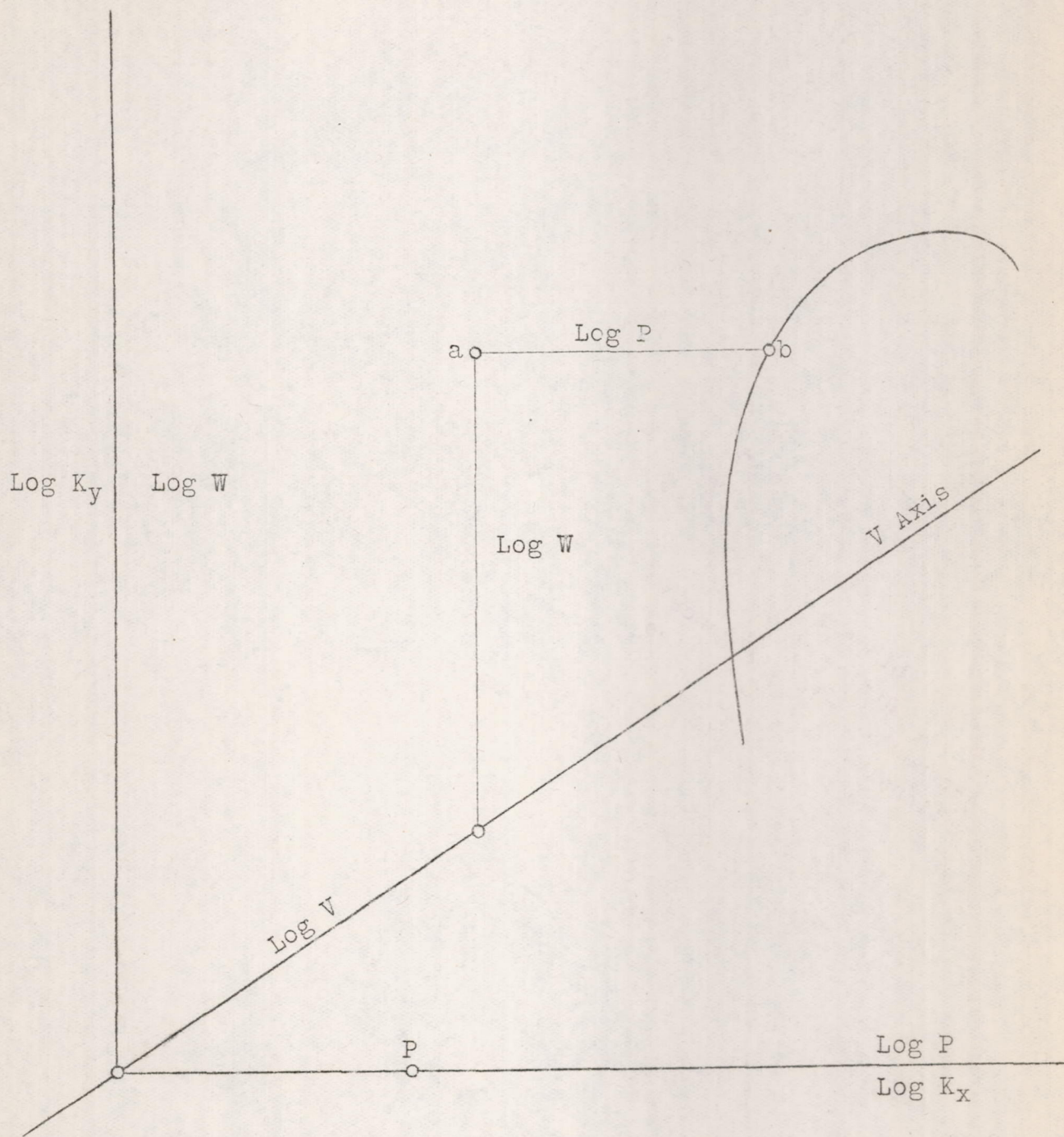


Fig.8

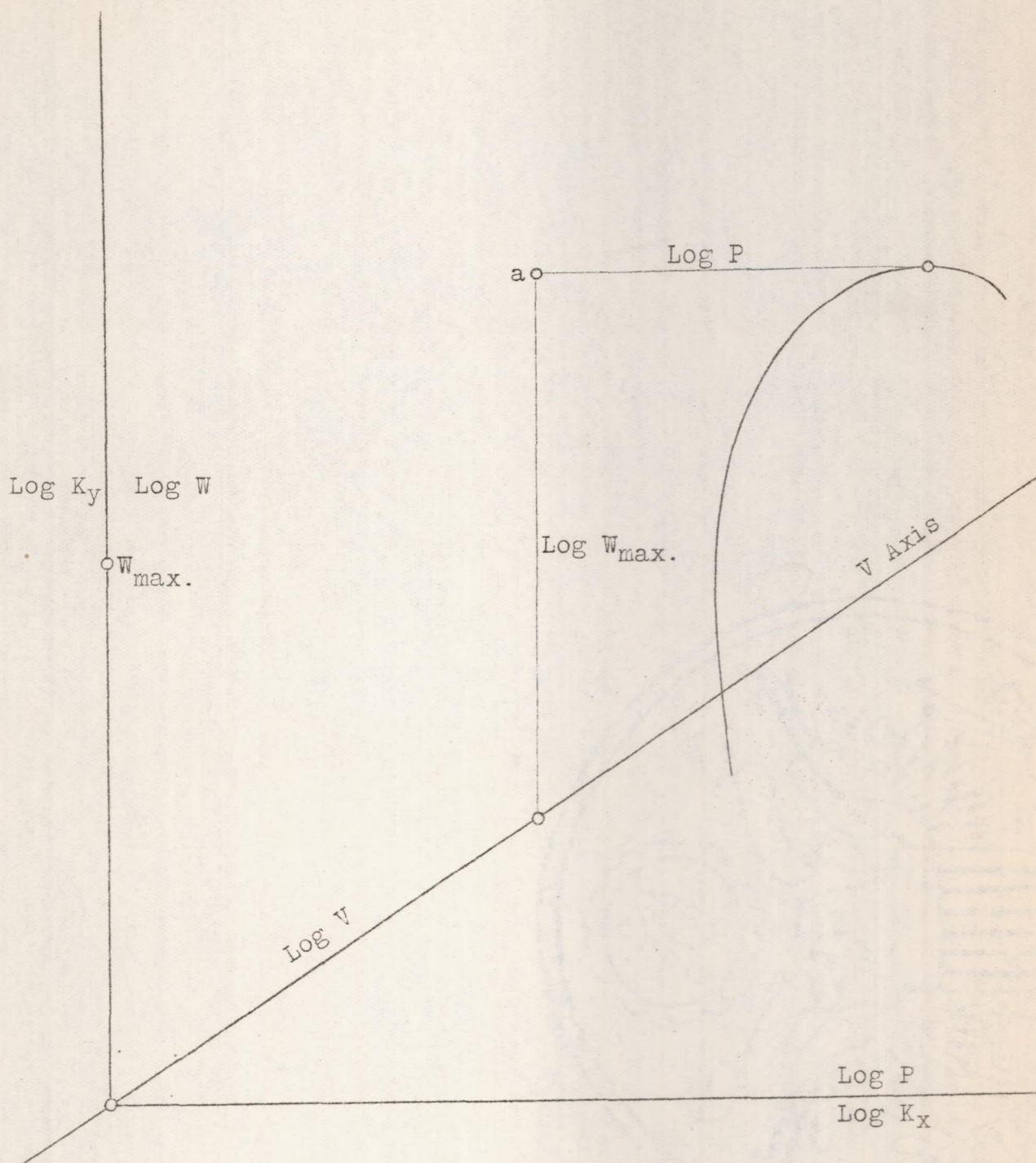


Fig.9

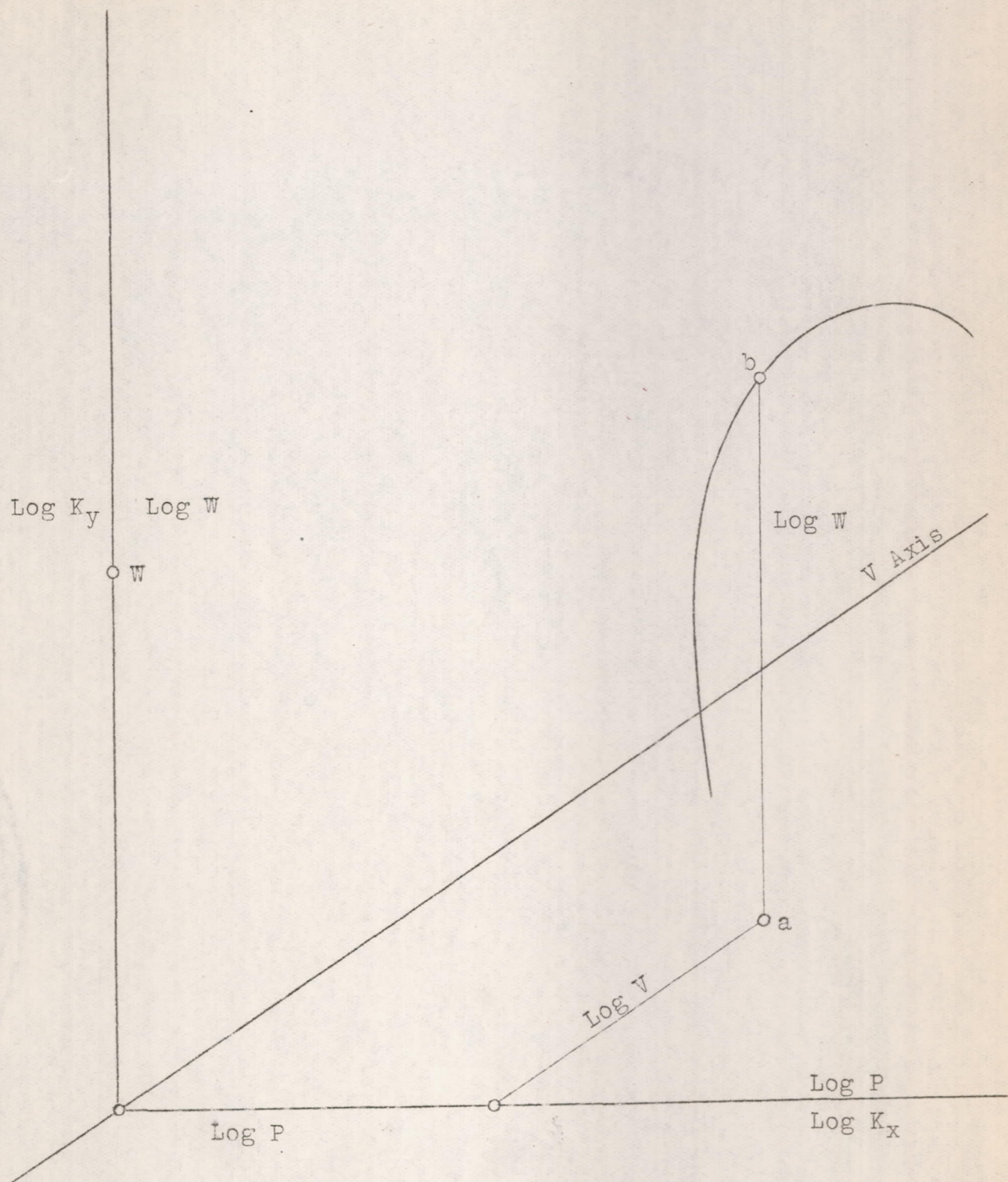


Fig.10

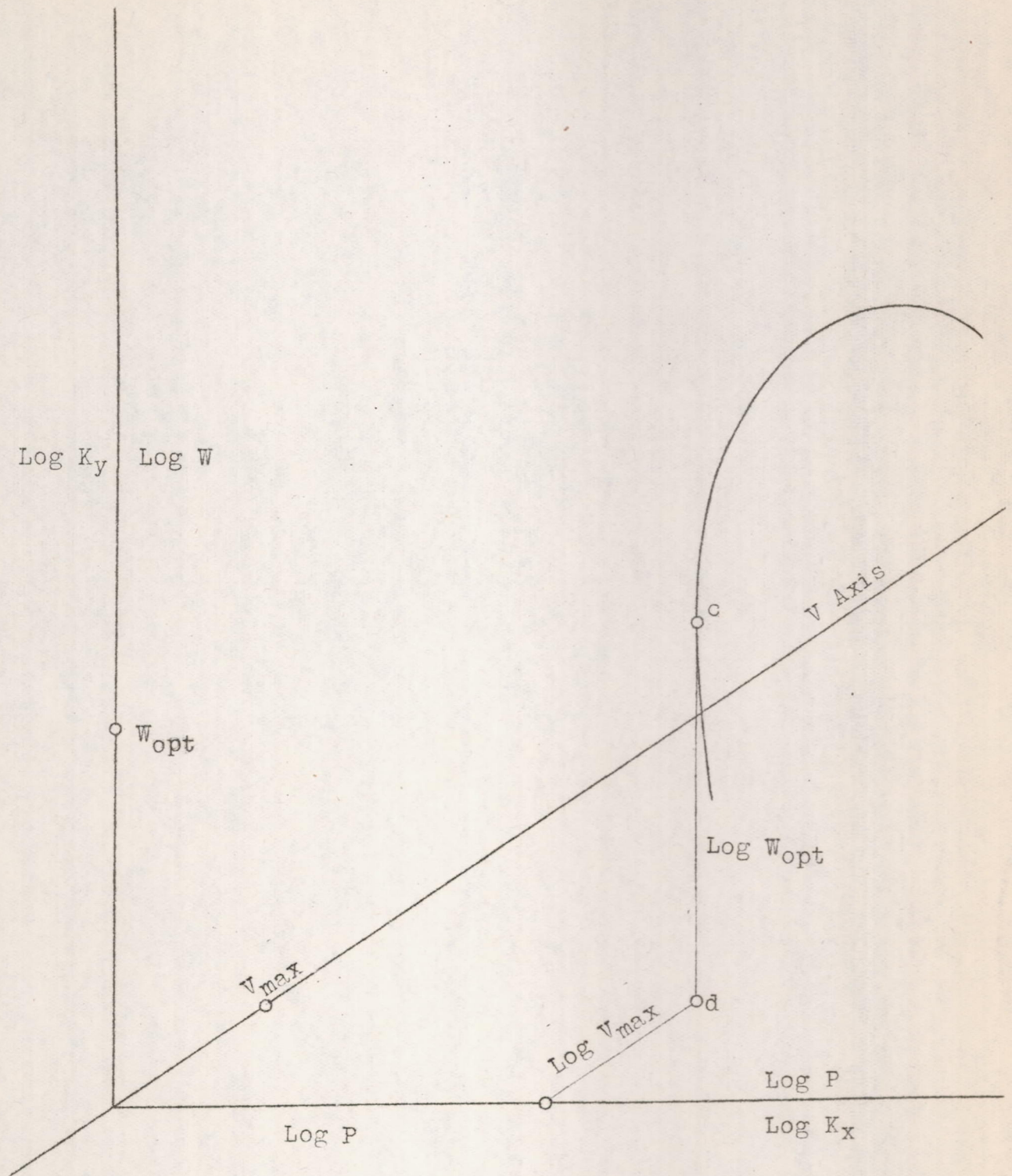


Fig.11

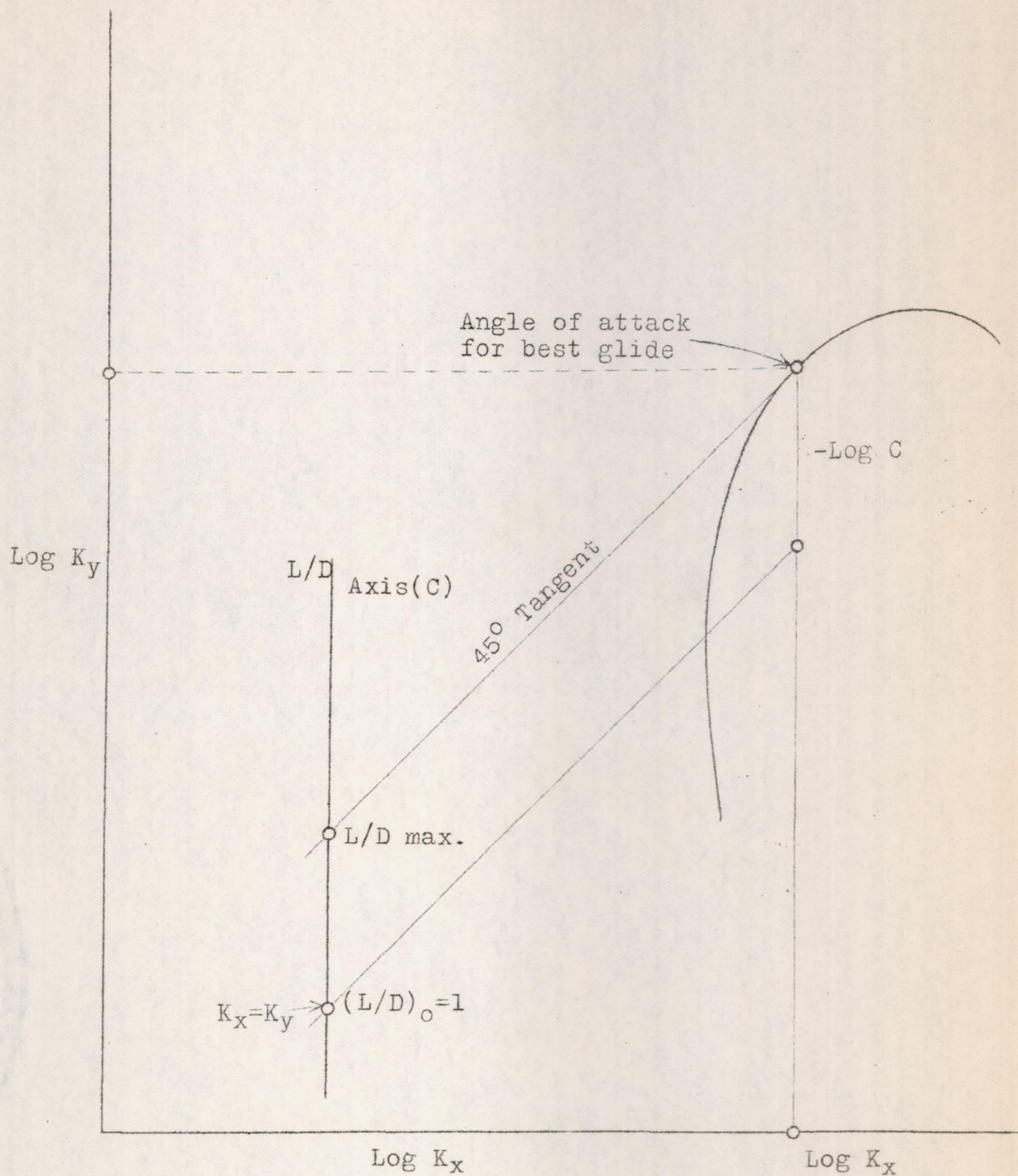


Fig.12

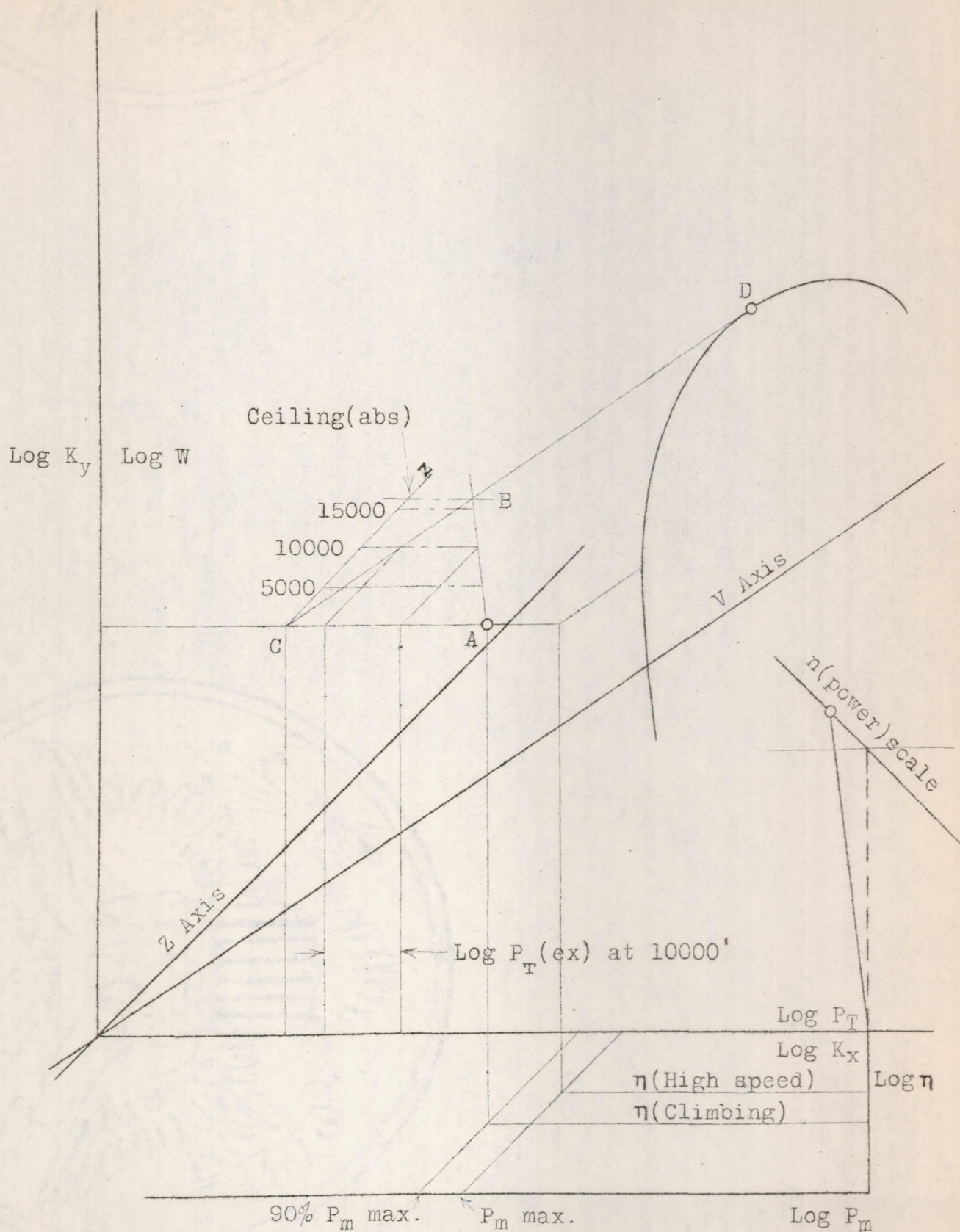


Fig.13

APPENDIX III

SUMMARY OF PERFORMANCE

	SEA LEVEL	5000 Ft.	10,000 Ft.	20,000 Ft.	30,000 Ft.
Maximum speed	(ab) 151.5mph	(cd) 158.0mph	(ef) 154.5mph	(gh) 146.5mph	(jk) 139.0mph
Minimum speed	(lm) 56.0mph	(nm) 61.2mph	(om) 65.0mph	(pm) 76.8mph	(jq) 91.1mph
Speed-best climb	(rx) 63.3mph	(sx) 69.3mph	(tx) 73.8mph	(ux) 86.2mph	(vx) 102.5mph
Rate of climb	2,220 fpm	2,190 fpm	1,675 fpm	873 fpm	151 fpm
Absolute ceiling	(rw) 32,100 ft				
Speed at ceiling	(yx) 107.5mph				
Service ceiling	(ri) 30,800 ft				
Max total weight	(zm) 11,200 lb				
Max L/D	(a) 9.4				

Note:-- For altitude performance, engine power is assumed to remain constant up to 5,000 ft and then vary as the density ratio to the 1.1 power

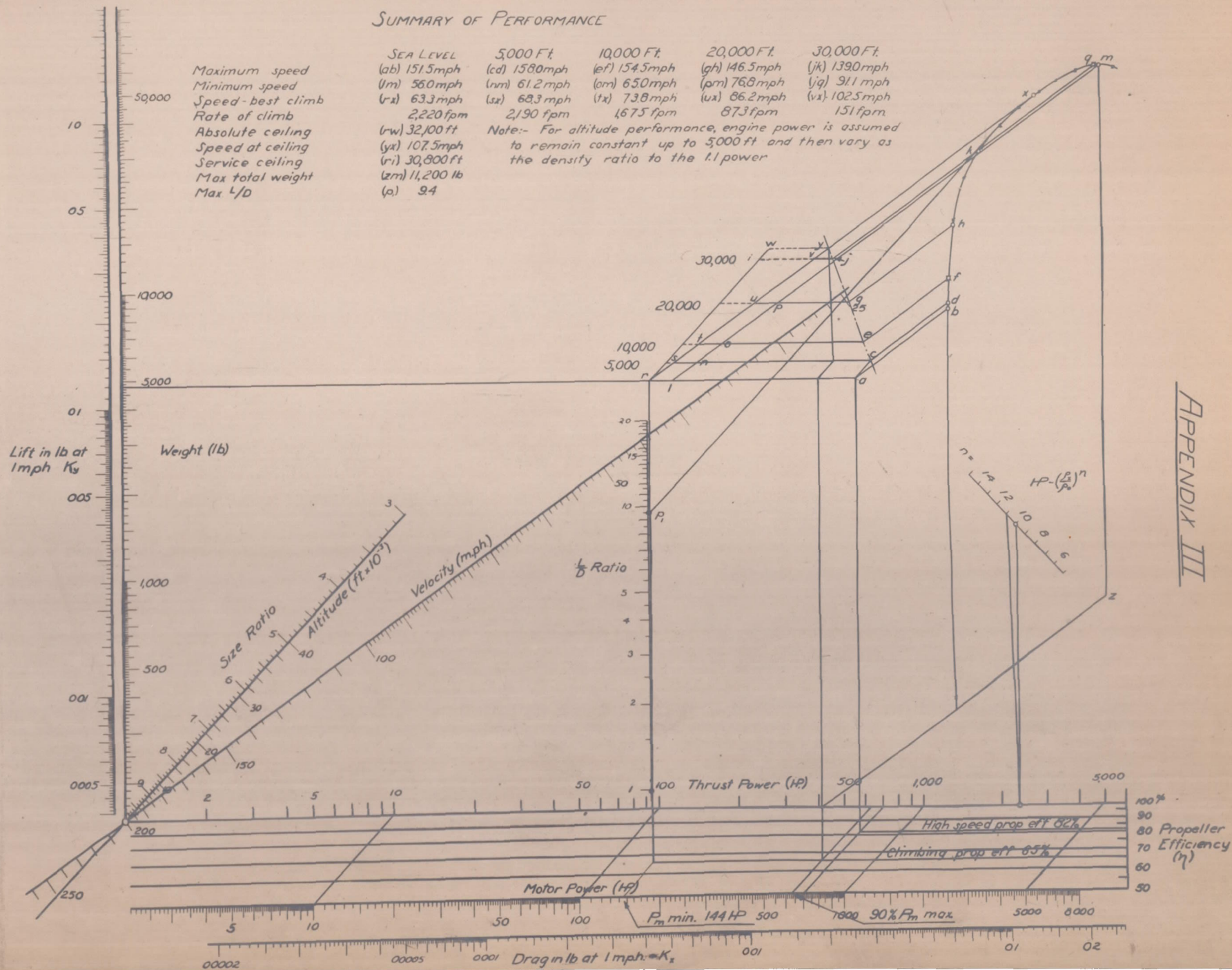


Fig. 14